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CONJUGATE FREE CONVECTION FROM HORIZONTAL, CONDUCTING CIRCULAR CYLINDERS

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Abstract—The concept of convective heat transfer is usually associated with specified definite boundary conditions at the interface between the heat dissipating or absorbing solid and the convecting fluid. Under many conditions arising in actual applications such an assumption may not be justified: the temperature and flux distribution on the interface link the temperature field in the solid to the convective motion and temperature of the fluid. This work considers some cases in which this coupling is of primary importance.

1. INTRODUCTION

THE CONCEPT of convective heat transfer between a solid (which may be heat dissipating or heat absorbent) and a fluid in contact with a part of its surface is usually specified through a well known set of governing equations, with *a priori* boundary conditions. In idealized cases this reduces to a statement that the interface is isothermal (corresponding to the qualitative assumption of infinite heat conductivity in the solid in comparison with that of the fluid). Alternatively the interface is one of constant or otherwise specified flux which implies negligible heat conduction in the solid. This *a priori* specification of boundary conditions leads to well-set problems for the two temperature fields (those in the solid and in the fluid) of the Dirichlet, Neumann or mixed type. However, very often these idealized conditions do not apply in actual physical situations, where the solid has in general neither vanishingly small nor infinitely large conductivity in comparison to that of the fluid. The two temperature distributions are then coupled and in some cases buoyancy effects lead to coupling with the equations of motion in the fluid as well. Occasionally corrections are introduced for this *a posteriori*.

Previous work on these coupled convective/conductive fields has been done by Perelman [1] for forced boundary-layer flow over the surface of a solid containing distributed heat sources and for convecting flow in a pipe by Genin [2] and Davis and Gill [3]. Other authors [4–6] have considered the particular influence of thermal coupling upon the performance of fluxmeters. In all these applications buoyancy effects are not involved and the flow

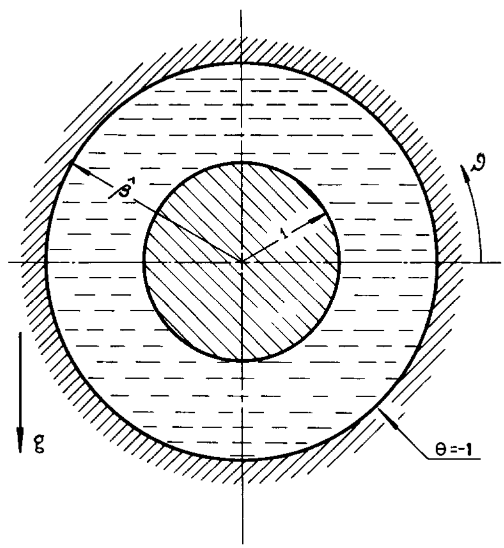

FIG. 1. Geometry of cylinders.

is specified independently of the temperature distribution. Of greater interest are those situations where the free convective effect is dominant: the convective motion itself is then caused by the temperature coupling effect. Such a case was investigated for example by Kelleher and Yang [7] for a well defined two-dimensional geometry (a downward projecting idealized fin) for which a solution was obtained in the form of a Goertler series expansion. and by Zinnes [8] for boundary-layer flow.

Here the case of steady heat transfer between two concentric infinitely long horizontal cylinders is considered. The inner core cylinder consists of a heat conducting solid which carries either a line source along its axis. distributed heat sourced or both. The core is surrounded by an essentially constant property fluid. Heat dissipation takes place by free convection alone. to the outer concentric cylindrical envelope which is assumed to be at a specified reference temperature or flux. Fig. 1. As there is no forced flow. the convective motion is driven by buoyancy. resulting in a coupling of great interest and complexity. The technological application of which this model is representative is the cooling of nuclear reactor fuel elements, for the case when the forced circulation system fails.

It will be shown that uniformly valid solutions for this case can be obtained only under serious restrictions. In particular the radius ratio of outer to inner cylinder has to remain finite: the single cylinder in an unbounded domain of fluid is thus excluded.

When the conductivity of the core cylinder material is infinitely large in comparison to that of the fluid then the interface between core and fluid will be isothermal. This particular case has already been discussed. e.g. Mack and Bishop [9]. The temperature rise of the inner cylinder surface will cause a warming of the adjacent fluid. Convective motion will then arise under even the slightest heating as the vector gravity is everywhere inclined to the solid to fluid interface. Once a quasi-steady state has been attained. the total heat flux gained or lost by the fluid at the inner boundary is compensated exactly by the exchange at the outer envelope of the fluid domain. In view of the fact that the motion is entirely buoyancy driven one would expect a solution to take the form of an asymptotic expansion in the parameter G. the Grashof number. Once the flow in the annular region has been well established, heat will be transported inside the solid to the parts of the envelope where the convective transport is most effective (this is *not* the uppermost location on the core cylinder). As the cylinder is assumed to be infinitely conductive this preferential flux distribution does not enable a calculation of the temperature distribution inside the solid. The flow arising outside the core evolves in rather characteristic patterns of single or multiple kidney-shaped cell configurations on each side of the vertical plane of symmetry. with a jet like flow polarization above the apex of the core. [10–12]. In actual physical situations the ratio of heat conductivities of solid to fluid is of course not infinitely large. Therefore the transport of heat inside the core to locations on the surface of greatest convective effect will be associated with a temperature gradient which is not negligibly small. An uneven core temperature distribution results, together with a rather complicated interplay with the resultant convective motion. The notion of ratio of thermal conductivities in the above should be understood in a qualitative sense as further parameters will enter into the analysis of the effect.

In the limit of zero thermal conductivity in the core. the interface becomes one of constant flux. The temperature distribution in the core is then again unaffected by convection.

In the present analysis the constancy of physical properties. except for the dependence of the density upon the temperature. and the validity of the Boussinesq approximations are assumed throughout. The method of analysis. the results and their physical interpretation will be discussed in detail.

2. ANALYSIS

The equations defining the fields of flow and temperature may be readily reduced to the dimensionless forms of the vorticity-transport equation and the energy equation as follows:

$$\nabla^4\psi = \frac{1}{r}\left[-\frac{\partial(\nabla^2\psi, \psi)}{\partial(r, \vartheta)} + G.\frac{\partial(\theta, r\cdot\sin\vartheta)}{\partial(r, \vartheta)}\right], \quad 1 \leqslant r \leqslant \hat{\beta} \quad (1)$$

$$\nabla^2\theta = \frac{\sigma}{r}\frac{\partial(\theta, \psi)}{\partial(r, \vartheta)}, \quad 1 \leqslant r \leqslant \hat{\beta} \quad (2)$$

and

$$\nabla^2\tilde{\theta} = A(r, \vartheta, \tilde{\theta}), \quad 0 \leqslant r \leqslant 1.† \quad (3)$$

Here r is the radial coordinate rendered dimensionless with the inner core radius for convenience; ϑ is the angular coordinate from the *horizontal*, $\hat{\beta}$ is the radius ratio of the outer to inner cylinder.

ψ is the dimensionless stream function. $\psi = \psi^*/\nu$, ψ^* is dimensional and ν is the kinematic viscosity of the fluid. $\nabla^2\psi$ is the dimensionless vorticity. G is a Grashof number defined as follows for the case of a single line source in the core.

$$G = \frac{g\beta a^3\dot{q}(\hat{\beta} - 1)^3}{2\pi K_{fl}\nu^2} \quad (4)$$

where g is the gravity const; β the expansion coefficient of the fluid at constant pressure; a the core radius; $\dot{q}$ the line-source strength per unit time; $\hat{\beta}$ the ratio of outer envelope to core radii; K the thermal conductivity. and the subscript "fl" refers to the fluid.

θ is the dimensionless temperature, conveniently normalized to yield zero on the core envelope for the fundamental-term solution of equations (2) and (3) when A, the source function, is zero. We find that

$$\Delta T_0 = \nabla T_{ref}{}^*\,\theta_0, \quad (5)$$

† In what follows the tilde symbol will refer to this region.

$$\Delta T_{ref} = \frac{\dot{q}}{2\pi K_{fl}}(\hat{\beta} - 1)^3. \quad (5a)$$

σ is the Prandtl number ν/κ of the fluid. where κ is the thermal diffusivity.

We shall retain the same reference temperature for the case of non-vanishing A as well. though the solution will of course be affected. For simplicity the outer envelope is assumed to be at -1 datum temperature. Then the boundary conditions which the equations (1)–(3) have to fulfill are as follows:

(i) Equality of temperatures at the core/fluid interface.

$$\theta|_{r=1} = \tilde{\theta}|_{r=1}. \quad (6)$$

(ii) In steady state, the equality of fluxes at that interface.

$$\left.\frac{\partial\theta}{\partial r}\right|_{r=1} = \omega\left.\frac{\partial\tilde{\theta}}{\partial r}\right|_{r=1}, \quad (7)$$

where ω is the ratio of core to fluid thermal conductivities.

(iii) The temperature at the outer envelope (which is assumed to possess infinitely large conductivity):

$$\theta|_{r=\hat{\beta}} = -1. \quad (8)$$

(iv) Vanishing of the stream function and its gradients on all solid boundaries.

$$\psi_{(r=1)} = \left.\frac{\partial\psi}{\partial r}\right|_{r=1} = \psi_{(r=\hat{\beta})} = \left.\frac{\partial\psi}{\partial r}\right|_{r=\hat{\beta}} \equiv 0. \quad (9)$$

2.1 *Expansion technique*

Through a trial-and-error procedure it is found that only the terms retained in the following expansions, ψ_i and θ_i (where i is an integer index) give rise to functions which do not vanish identically:

$$\psi = +G\psi_1 + G^2\psi_2^a + \sigma G^2\psi_2^b + G^3\psi_3^a + \sigma G^3\psi_3^b + \sigma^2 G^3\psi_3^c + \ldots \quad (1 \leqslant r \leqslant \hat{\beta}), \quad (10)$$

and

$$\theta = \theta_0 + R\theta_1 + RG\,\theta_2^a + R^2\theta_2^b + R^2G\,\theta_3^a + \ldots \quad (11)$$

where R is the Rayleigh number, $R = \sigma.G$.

For all other combinations of coefficients σ and G the associated functions vanish identically. Now, it is immediately apparent that whereas the expansion for the temperature θ possesses a zeroeth-order term, representing conduction alone between core and envelope, the expansion for the stream-function ψ starts with a 1st-order term in the expansion parameter G. Moreover, an interesting point is that the power exponent of σ in any term is at most of the order of that of G. Therefore, expanding in the Grashof number rather than equivalently in terms of the Rayleigh number R as adopted in [9], would seem to present here some slight advantage. This is, however, not so for fluids having a very large value of the Prandtl number σ.

The first three terms in the asymptotic expansion (10) and the first four in (11) were determined explicitly and will be given below. As the expansion proceeds the algebraical expressions for the various terms become rather cumbersome. Details of the interaction between the various parameters are lost in the higher order terms, so that no additional physical insight is gained by the inclusion of further terms.

The asymptotic expansions (10) and (11) represent perturbations on a first term which represents a creeping-flow solution, as may easily be verified through insertion into the equations of motion. Therefore this regular expansion must ultimately diverge for sufficiently large values of G whatever the value of σ may be. Happily, as already found by Mack and Bishop [9] the behaviour of the functions associated with the various terms is such that divergence does not take place until G is of order 10^4 (for $\sigma = 1$). As the expansion is around a Stokes type first order term it is necessary to point out that it should not be expected to be uniformly valid over the complete range of $\hat{\beta}$ from 1 to ∞. Proudman and Pearson [13], Hieber and Gebhard [14].

The first term in the expansion (11) is determined from,

$$\nabla^2 \theta_0 = A \quad (0 < r \leqslant 1) \tag{12}$$

where for the isolated line source, $A = 0$ $(r \neq 0)$. We then obtain

$$\theta_0 = -\ln r/\ln\hat{\beta} \quad (1 \leqslant r \leqslant \hat{\beta}) \tag{13}$$

and

$$\tilde{\theta}_0 = -\ln r/(\omega \ln\hat{\beta}) \quad (0 < r \leqslant 1). \tag{14}$$

where the tilde superscript refers to the core material as usual and ω is the ratio of conductivities. *core to fluid.*

For the case of a constant non-zero A we find,

$$\theta_0 = -B\ln r/\ln\hat{\beta} \quad (1 \leqslant r \leqslant \hat{\beta}). \tag{15}$$

$$\tilde{\theta}_0 = -\frac{1-r^2}{2\omega(\hat{\beta}-1)} - \frac{1}{\omega}\left(\frac{B}{\ln\hat{\beta}} + \frac{1}{\hat{\beta}-1}\right).\ln r \quad (0 < r \leqslant 1). \tag{16}$$

where

$$B = 1 + \frac{\dot{q}}{a^2 A}. \tag{17}$$

The case of a core with a central, concentric bore has also been solved but will not be given here as it adds little but algebraic complication.

The first term in (10) may now be determined from,

$$\nabla^4\psi_1 = \frac{\partial\theta_0}{\partial r}\cos\vartheta. \tag{18}$$

For convenience we shall proceed for the sole line-source. We find,

$$\psi_1 = [A_1 r^3 + B_1 r\ln r + C_1 r + D_1/r - r^3\ln r]\frac{\cos\vartheta}{16\ln\hat{\beta}} \quad (1 \leqslant r \leqslant \hat{\beta}). \tag{19}$$

where

$$A_1 = -\{\hat{\beta}^4[2\ln\hat{\beta}(1-2\ln\hat{\beta})+1] + (1 + 2\ln\hat{\beta})\cdot(1-2\hat{\beta}^2)\}/\mathrm{DEN}_1 \tag{20}$$

$$B_1 = -2\{\hat{\beta}^6 - \hat{\beta}^4\cdot(1+4\ln\hat{\beta}) - \hat{\beta}^2(1-4\ln\hat{\beta}) + 1\}/\mathrm{DEN}_1 \tag{21}$$

$$C_1 = \{\hat{\beta}^6 - \hat{\beta}^4[2\ln\hat{\beta}(4\ln\hat{\beta}-1)+1] - \hat{\beta}^2(1+4\ln\hat{\beta}) + (1+2\ln\hat{\beta})\}/\mathrm{DEN}_1 \tag{22}$$

$$D_1 = -\hat{\beta}^2\{\hat{\beta}^4 - 2\hat{\beta}^2(1 + 2\ln^2\hat{\beta}) + 1\}/\mathrm{DEN}_1. \quad (23)$$

The denominator expression DEN_1 is,

$$\mathrm{DEN}_1 = 4\hat{\beta}^4(\ln\hat{\beta} - 1) + 2\hat{\beta}^2 - (1 + \ln\hat{\beta}). \quad (24)$$

Thus, to first order there is (as may be expected) no difference between the conjugate and non-conjugate cases.

For the first correction term to the temperature we find the equation.

$$\nabla^2\theta_1 = \left\{A_1 r^3 + B_1 r\ln r + C_1 r + \frac{D_1}{r} - r^3\ln r\right\}\cdot\frac{\sin\vartheta}{(4r\ln\hat{\beta})^2} \qquad (1 \leqslant r \leqslant \hat{\beta}). \quad (25)$$

and the solutions,

$$\theta^2 = \frac{1}{128}\left\{-r^3\ln r + \left(A_1 + \frac{3}{4}\right)r^3 + + 2B_1 r\ln r(\ln r - 1) + 4C_1 r\ln r - 4D_1\frac{\ln r}{r} + 8E_1 r + \frac{8F_1}{r}\right\}\frac{\sin\vartheta}{\ln^2\hat{\beta}} \qquad (1 \leqslant r \leqslant \hat{\beta}) \quad (26)$$

$$\tilde{\theta}_1 = \frac{\tilde{E}_1}{16}r\frac{\sin\vartheta}{\ln^2} \qquad (0 < r \leqslant 1). \quad (27)$$

The constants of integration of the complementary solutions are given by,

$$E_1 = \{-P_2 - \hat{\beta}^2(1 + \omega)\cdot P_3 + \omega P_1\}/(\hat{\beta}^2\,\mathrm{DEN}_2) \quad (28)$$

$$F_1 = \{-\omega P_1 + P_2 - (1 - \omega)P_3)\}/\mathrm{DEN}_2 \quad (29)$$

$$\tilde{E}_1 = \{(1 - 1/\hat{\beta}^2)\cdot P_2 + (1 + 1/\hat{\beta}^2)P_1 - 2P_3\}/\mathrm{DEN}_2 \quad (30)$$

with

$$\mathrm{DEN}_2 = -[\omega(1 - \hat{\beta}^{-2}) + 1 + \hat{\beta}^{-2}] \quad (31)$$

and

$$P_1 = -(A_1 + \tfrac{3}{4})/8$$
$$P_2 = (D_1 - C_1 + B_1/2)/2 - 3A_1/8 - 5/32$$
$$P_3 = \tfrac{1}{2}(\hat{\beta}/2)^2\cdot(\ln\hat{\beta} - A_1 - \tfrac{3}{4}) + (B_1/4) \times \ln\hat{\beta}(1 - \ln\hat{\beta}) - \tfrac{1}{2}(C_1 - D_1\cdot\hat{\beta}^{-2})\ln\hat{\beta}.$$

The next expansion term in (10) is now obtained. It will be the first term to impose angular dissymmetry on the resultant solution.

The equation is,

$$\nabla^4\psi_2^a = \frac{1}{2\times 16^2}\cdot\{16r^2\ln^2 r - 4(8A_1 - 3)r^2\ln r - 8B_1\ln r + 4(B_1^2 + 4D_1)(\ln r)/r^2 + 2(3 - 6A_1 - 8A_1^2)\cdot r^2 + 8(2A_1B_1 - B_1 + C_1) + 2(10D_1 - 8A_1D_1 + B_1^2 + 2B_1C_1)r^{-2}\}\sin 2\vartheta/\ln^2\hat{\beta}. \quad (32)$$

The solution is found quite straightforwardly as follows,

$$\psi_2^a = \frac{1}{3\times(16)^3}\left\{r^6\cdot\left[\ln r\cdot\left(\ln r - 2A_1 - \frac{4}{3}\right) + A_1\cdot\left(A_1 + \frac{4}{3}\right) + \frac{149}{144}\right] + r^4\cdot\ln r\cdot\left[-2B_1 \times\ln r + B_1\times\left(8A_1 - \frac{1}{3}\right) + 4C_1\right] - \frac{3}{2}r^2\ln r \times[2(B_1^2 + 4D_1)\ln r + B_1\times(B_1 + 4C_1) + 16D_1\cdot(1 - A_1)] + 24(\mathscr{A}_2^a r^4 + \mathscr{B}_2^a + \mathscr{C}_2^a r^2 + \mathscr{D}_2^a r^{-2})\right\}\frac{\sin 2\vartheta}{\ln^2\hat{\beta}}. \quad (33)$$

The constants $\mathscr{A}_2^a$, $\mathscr{B}_2^a$, $\mathscr{C}_2^a$ and $\mathscr{D}_2^a$ are those of the complementary solution and must be determined to fulfill the four boundary conditions upon ψ. They can be found by inverting a 4×4 matrix, or even in closed form. The determination of the term σG^2 is somewhat simpler. The equation is,

$$\nabla^4\psi_2^b = \frac{1}{16^2}\left\{r^2\left(2A_1 + \frac{1}{2} - 2\ln r\right) + 2B_1(2\ln r - 1) + 4C_1 + \frac{4D_1}{r^2} \times(2\ln r - 1) - \frac{16F_1}{r^2}\right\}\sin 2\vartheta/\ln^2\hat{\beta}. \quad (34)$$

The solution is,

$$\psi_2^b = \frac{1}{2\times(16)^3}\left\{\frac{r^6}{6}\left(A_1 + \frac{31}{24} - \ln r\right)\right.$$

$$+\frac{r^4}{3}\ln r[4B_1\left(\ln r-\frac{17}{6}\right)+8C_1]$$
$$-4r^2\ln r\,[D_1(2\ln r-3)-8F_1]$$
$$\left.+8(\mathscr{A}_2^b r^4+\mathscr{B}_2^b+\mathscr{C}_2^b r^2+\mathscr{D}_2^b r^{-2})\right\}\frac{\sin 2\vartheta}{\ln^2\beta}. \quad (35)$$

The constants $\mathscr{A}_2^b$ through $\mathscr{D}_2^b$ are again determined as described before, following equation (33).

It is now possible to proceed to the determination of the higher-order temperature terms. The equation for the term in RG, equation (11), reduces to

$$\nabla^2\theta_2^a=\frac{-1}{3\times(16)^3}\left\{r^4\left[\ln r\left(\ln r-2A_1-\frac{4}{3}\right)\right.\right.$$
$$\left.+A_1\left(A_1+\frac{4}{3}\right)+\frac{149}{144}\right]+r^2\ln r\left[-2B_1\ln r\right.$$
$$\left.+B_1\left(8A_1+\frac{1}{3}\right)+4C_1\right]-\frac{3}{2}\ln r\times[2(B_1^2$$
$$+4D_1)\ln r+B_1(B_1+4C_1)+16D_1(1-A_1)]$$
$$\left.+24(\mathscr{A}_2^a r^2+\mathscr{B}_2^a r^{-2}+\mathscr{C}_2^a+\mathscr{D}_2^a r^{-4})\right\}$$
$$\times\frac{2\cos 2\vartheta}{\ln^3\beta}. \quad (36)$$

The solution is found to be, after some rearrangement

$$\theta_2^a=\frac{-2}{3\times(16)^3}\left[\!\!\left[\frac{r^6}{32}\left\{\ln^2 r-\frac{(25/3)+8A_1}{4}\ln r\right.\right.\right.$$
$$\left.\left.+\frac{1}{8}\left[\frac{505}{36}+\frac{50}{3}A_1+8A_1^2\right]\right]\right\}+\frac{r^4}{3}\left\{-\frac{B_1}{2}\ln^2 r\right.$$
$$+\left[\frac{B_1}{12}(7+24A_1)+C_1\right]\ln r-\frac{1}{3}$$
$$\left.\times\left(\frac{11}{12}B_1+4A_1B_1+2C_1\right)+6\mathscr{A}_2^a\right\}$$
$$-\frac{r^2}{4}\left\{(B_1^2+4D_1)\ln^3 r+3[B_1C_1\right.$$
$$+D_1(3-4A_1)]\ln^2 r-\frac{3}{2}[B_1C_1$$
$$\left.+D_1(3-4A_1)]\ln r-24\mathscr{C}_2^a\ln r+4\mathscr{E}_2^a\right\}$$
$$\left.\left.-6(\mathscr{B}_2^a+\mathscr{D}_2^a)\frac{\ln r}{r^2}\;-\mathscr{F}_2^a r^{-2}\right]\!\!\right]$$
$$\times\frac{\cos 2\vartheta}{\ln^3\beta}\quad(1\leqslant r\leqslant\beta). \quad (37)$$

The unknown constants here are just E_2^a and F_2^a, respectively $\tilde{E}_2^a$ and $\tilde{F}_2^a$ for the core cylinder. In view of the singularity of exponent 2 which $\tilde{F}_2^a$ would give rise to, we may immediately set $\tilde{F}_2^a\equiv 0$. Therefore, the expression for $\tilde{\theta}_2^a$ reduces to

$$\tilde{\theta}_2^a=-\frac{2}{3*(16)^3}\tilde{E}_2\frac{r^2\cos 2\vartheta}{\ln^3\beta}$$
$$(0\leqslant r\leqslant 1). \quad (38)$$

The last term of the expansion calculated is θ_2^b. It forms the solution of,

$$\nabla^2\theta_2^b=\frac{r^4}{(16)^3}\left[\!\!\left[\frac{1}{6}\left(\ln r+\frac{77}{24}-A_1\right)\times\cos 2\vartheta\right.\right.$$
$$-6\ln^2 r+2\left(6A_1+\frac{5}{4}\right)\ln r+\frac{3}{4}$$
$$\left.\left.-2A_1\left(3A_1+\frac{5}{4}\right)\right]\!\!\right](\ln\beta)^{-3}$$
$$+\frac{r^2}{(16)^3}\left[\!\!\left[4B_1\ln^3 r-4\left(\frac{7}{3}B_1+A_1B_1\right.\right.\right.$$
$$\left.-2C_1\right)\ln^2 r+5\left(2A_1+\frac{19}{18}\right)B_1\ln r$$
$$-2C_1\left(\frac{7}{3}+4A_1\right)\ln r+16E_1\ln r$$
$$-B_1\left(3A_1+\frac{3}{4}\right)+C_1\left(6A_1+\frac{1}{2}\right)$$
$$\left.-8E_1(2A_1-1)-8\mathscr{A}_2^b\right]\times\cos 2\vartheta$$
$$+8B_1\ln^3 r-8\left[B_1\left(A_1-\frac{1}{4}\right)-2C_1\right]\ln^2 r$$
$$-16\left[\left(A_1-\frac{3}{4}\right)C_1-2E_1+\frac{5}{16}B_1\right]\times\ln r$$
$$+B_1\left(A_1-\frac{3}{4}\right)-2C_1(1+4A_1)$$

$$+ 8E_1(1 - 4A_1)\Big]\Big] (\ln \hat{\beta})$$
$$+ \frac{r^0}{(16)^3}\Big[\Big[\{2(B_1^2 - 4D_1)\ln^2 r + 4[B_1C_1 - 4D_1(1 - A_1)] \times \ln r + 2[2C_1^2 - B_1(4E_1 + C_1) + 4F_1 + D_1 - 16A_1F_1 - 4\mathscr{C}_2^b]\}$$
$$\times \cos 2\vartheta + 2\Big\{- 2B_1^2 \ln^3 r - [B_1(B_1 + 6C_1) + 4D_1]\ln^2 r + [2B_1(B_1 - 2C_1 - 4E_1) - 4C_1^2 + D_1(4A_1 - 3) + 8F_1]\ln r$$
$$+ \Big[- 4E_1(2C_1 + B_1) + 2A_1\left(\frac{D_1}{2} - 4F_1\right) - 2C_1^2 - \frac{D_1}{4} + 4F_1 + B_1C_1\Big]\Big\}\Big]\Big] (\ln \hat{\beta})^{-3}$$
$$+ \frac{r^{-2}}{(16)^3}\Big[\Big[8\Big\{+ \frac{D_1}{2}(3B_1 \ln^2 r + 4C_1 \ln r + 4E_1) - F_1(2B_1 \times \ln r + B_1 + 2C_1) - \frac{D_1B_1}{2} - \mathscr{B}_2^b\Big\} \cos 2\vartheta + \Big\{8 - F_1B_1 + \frac{B_1D_1}{2}\left(\ln r + \frac{1}{2}\right)\Big\}\Big]\Big] (\ln \hat{\beta})^{-3}$$
$$+ \frac{r^{-4}}{(16)^3}\Big[- 4(D_1^2 + 2\mathscr{D}_2^b) \cos 2\vartheta + \left(1 + \frac{4F_1}{D_1} - 2 \ln r\right) \times (2D_1)^2\Big] (\ln \hat{\beta})^{-3}. \quad (39)$$

The solution is obtained in the standard way to yield,

$$\theta_2^b = (16 \ln \hat{\beta})^{-3}\Big[\Big[\frac{r^6}{6}\Big\{\frac{1}{32}\left(\ln r + \frac{17}{6} - A_1\right) \times \cos 2\vartheta + \Big[\ln r \times \left(- \ln r + \frac{13}{12}\right) - \frac{13}{72} + A_1 \times \left(2 \ln r - A_1 - \frac{13}{12}\right)\Big]\Big\} + \frac{r^4}{3}\Big\{B_1\Big[\ln^3 r$$
$$- \frac{13}{12}\ln^2 r + \frac{475}{72}\ln r - \frac{1669}{432}\Big] + C_1 \cdot \Big[2 \ln^2 r - \frac{23}{6}\ln r + \frac{169}{72}\Big] + A_1B_1 \times \Big[- \ln^2 r + \frac{23}{6}\ln r - \frac{13}{36}\Big] + A_1C_1 \times \Big[-2 \ln r + \frac{17}{6}\Big] + E_1$$
$$\times \Big[4 \ln r - \frac{2}{3}\Big] - 4E_1A_1 - 2\mathscr{A}_2^b\Big\} \cos 2\vartheta$$
$$+ \frac{r^4}{2}\Big\{B_1\Big[\ln^3 r - \frac{5}{4}\ln^2 r + \frac{1}{4}\ln r - \frac{1}{16}\Big] + C_1\Big[2 \ln^2 r + \frac{1}{2}\ln r - \frac{1}{4}\Big] + A_1B_1 \times \Big[- \ln^2 r + \ln r - \frac{1}{4}\Big] - 2A_1C_1 \ln r + E_1(4 \ln r - 4A_1 - 1)\Big\}$$
$$+ \frac{r^2}{2}\Big\{B_1\Big[B_1\left(\frac{\ln^2 r}{3} - \frac{\ln r}{4} + \frac{1}{8}\right) + C_1\left(\ln r - \frac{3}{2}\right) - 4E_1\Big] + D_1 \times \Big[- \frac{4}{3}\ln^2 r - 3 \ln r + \frac{5}{2} + 2A_1(2 \ln r - 1)\Big] + 2 \cdot \Big[C_1^2 + 2F_1 \times (1 - 4A_1) - 2\mathscr{C}_2^b\Big]\Big\} \ln r$$
$$\times \cos 2\vartheta + \frac{r^2}{2}\Big\{B_1[B_1(-2 \ln^3 r + 5 \ln^2 r - 5 \ln r - 5/2) + 2C_1(-3 \ln^2 r + 4 \ln r - 2) + 4E_1(-2 \ln r + 1)] + D_1\Big[-4 \ln^2 r$$
$$+ 5 \ln r - \frac{13}{4} + A_1(4 \ln r - 3)\Big] + 2C_1[C_1 \times (-2 \ln r + 1) - 4E_1] + 4F_1(2 \ln r - 1 - 2A_1)\Big\} + r^0\{2F_1(2B_1 \ln r + B_1 + 2C_1) - D_1 \times (3B_1 \ln^2 r + 4C_1 \ln r + 4E_1 + B_1) + 2\mathscr{B}_2^b\} \cos 2\vartheta$$
$$+ r^0\Big\{\frac{2}{3}B_1D_1 \ln^2 r\left(\ln r + \frac{3}{2}\right) - 4F_1B_1 \ln^2 r\Big\}$$
$$+ r^{-2} \times \Big[(D_1^2 + 2\mathscr{D}_2^b) \ln r \cos 2\vartheta$$

$$- D_1^2\left(2\ln r + 1 - \frac{4F_1}{D_1}\right)\Bigg] + (\mathscr{E}_2^b r^2 + \mathscr{F}_2^b r^{-2}) + \mathscr{G}_2^b + \mathscr{H}_2^b \ln r\Bigg]\Bigg] \qquad (1 \leqslant r \leqslant \hat{\beta}). \quad (40)$$

For the solid one has.

$$\theta_2^b = (16\ln\hat{\beta})^{-3}\,[(\tilde{\mathscr{E}}_2^b r^2 + \tilde{\mathscr{F}}_2^b r^{-2})\cos 2\vartheta + \tilde{\mathscr{G}}_2^b + \tilde{\mathscr{H}}_2^b \ln r] \qquad (0 \leqslant r \leqslant 1). \quad (41)$$

As a singularity of order 2 is not permissible in the core. we may immediately set $\tilde{F}_2^b$ to zero. Also. from the temperature matching conditions at the interface we see that only one of the constants. G_2^b or $\tilde{G}_2^b$ need be retained. without loss of generality. We shall here retain $\tilde{G}_2^b$ and set G_2^b to zero.

Therefore.

$$G_2^b = F_2^b = 0. \qquad (42)$$

The increasing complexity and high-order asymmetry of the higher-order correction terms is now quite apparent. Moreover. as invariably happens when analytical series expansions are to be exploited numerically. convergence becomes slow and accuracy problematical. No higher-order terms will therefore be given.

3. NUMERICAL EVALUATION

The series expansions (10) and (11) were evaluated numerically for a variety of combinations of the physical parameters $\hat{\beta}$. ω. G and σ. Plots of streamlines. isotherms and isovorticity lines were obtained. As these plots look qualitatively similar for all physically reasonable combinations of the parameters $\hat{\beta}$ and ω (within the range of G and σ for which convergence to the order calculated is obtained) only one example of the plots is given here. Figs. 2–5. The characteristic cell formation already observed previously [9] on the case of an isothermal core is qualitatively reproduced: Two kidney-shaped circulatory systems fill the annular fluid region symmetrically to the vertical. Their centroids are located somewhat above the horizontal centre line of the cylinders.

The isothermals are continuous over the fluid/solid interface. but due to the conjugate nature of the problem they are "refracted". The lower the conductivity ratio ω the greater the alteration of the slope at the interface. as per boundary condition (7) above. The core isothermals are also no longer concentric around the geometrical center of the core.

A feature of the previous results cited had been the discovery of *multiple* cells on each side, under certain conditions. It is believed that as far as an analysis of the present type (and its computational evaluation) is concerned such multiple cell formation cannot form a genuine feature of either the conjugate or the non-conjugate cases. For as long as the terms retained in the asymptotic expansion converge. the first term must remain dominant. However. should multiple cells occur on each side. adjacent cells would have to show values of the stream function of *opposite* algebraical sign. As the dominant term is here *always* negative this can only mean that higher order terms have become larger than the fundamental term. Whenever therefore computer plots tended to show such features this was taken as evidence of divergence and the results were discarded. It is of course known that multiple cell formation can be observed experimentally: direct numerical integration of the partial differential equations would have to be employed in order to detect these analytically. It is of interest to note that the multicellular structure postulated by previous workers, e.g. [9–12, 15] is not found upon numerical integration of the 'linearized' equations of vorticity transport and energy performed by the same investigators [16] at values of the parameters where multiple cells had been shown experimentally to exist.

It is interesting to note on the plots of the results of the computations that the $\theta = 0$ curve no longer coincides with the core envelope. but penetrates the solid core at the lower elevation, and the fluid space at the top. This is of course the result of the "conjugateness" of the problem under consideration.

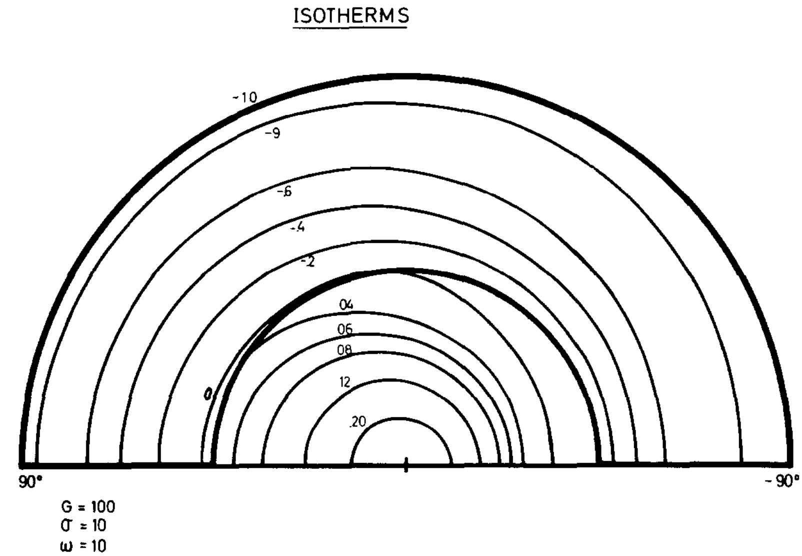

FIG. 2a. Isothermals for $G = 100$, $\sigma = 0{\cdot}72$ and $\omega = 10$.

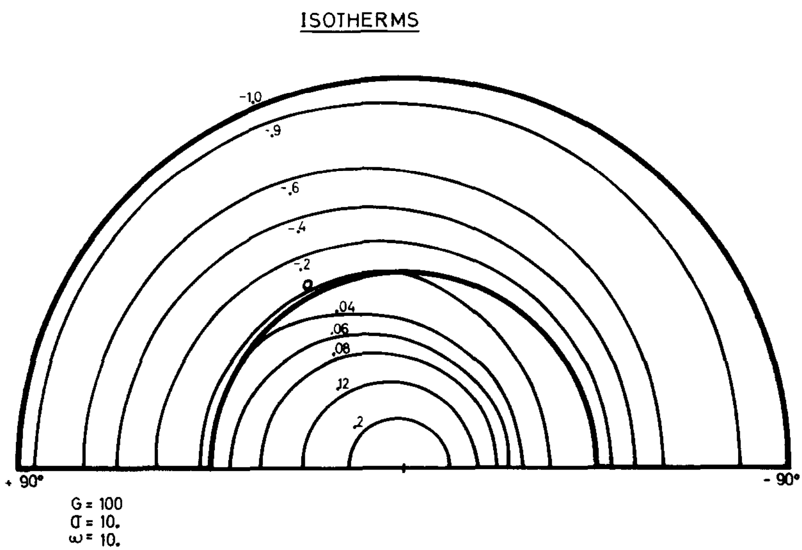

FIG. 2b. Iso-vorticity lines for $G = 100$, $\sigma = 0{\cdot}72$ and $\omega = 10$.

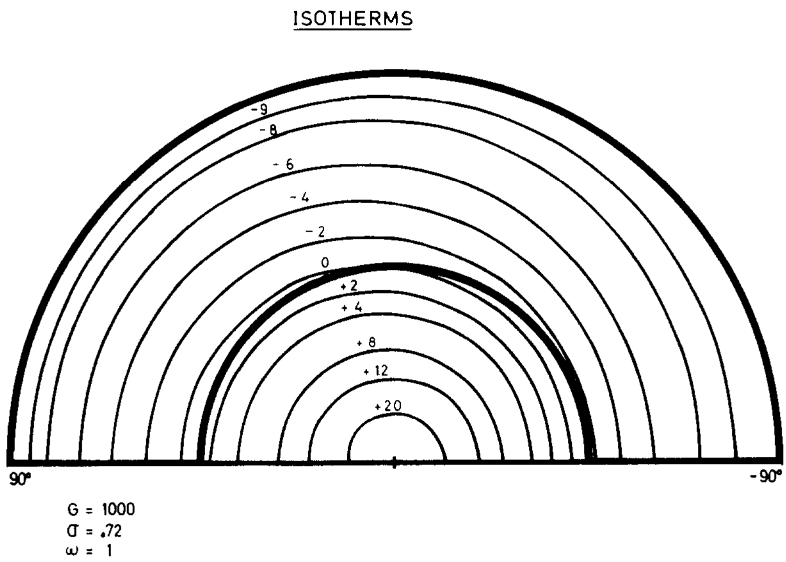

FIG. 3a. Isothermal lines for $G = 1000$, $\sigma = 0.72$ and $\omega = 1$.

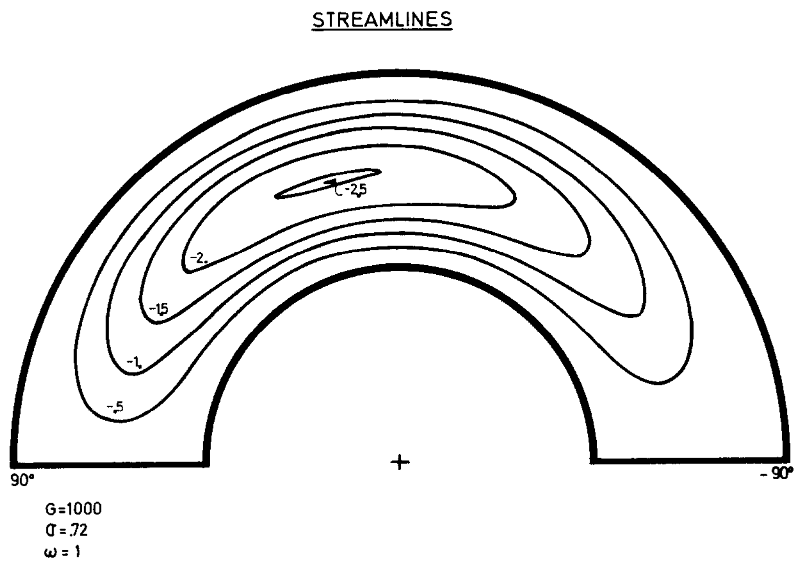

FIG. 3b. Streamlines for $G = 100$. $\sigma = 0.72$ and $\omega = 1$.

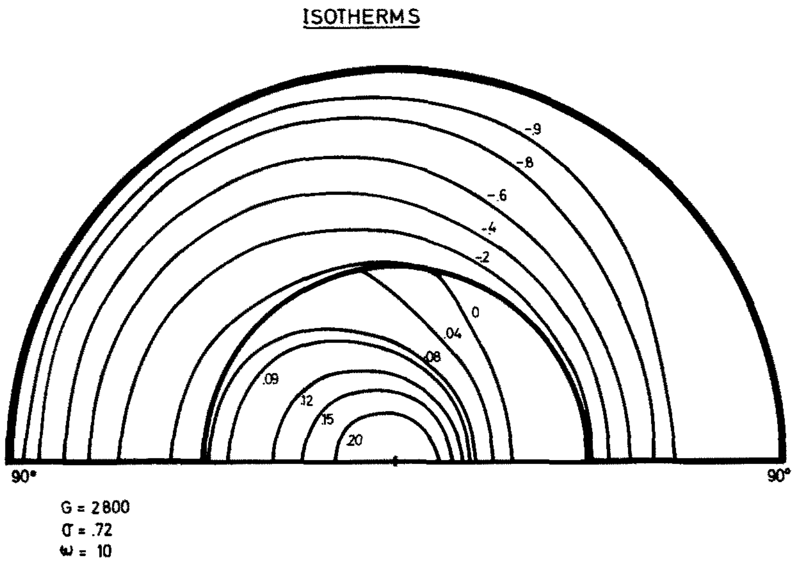

FIG. 4a. Isothermal lines for $G = 2800$, $\sigma = 0{\cdot}72$ and $\omega = 10$.

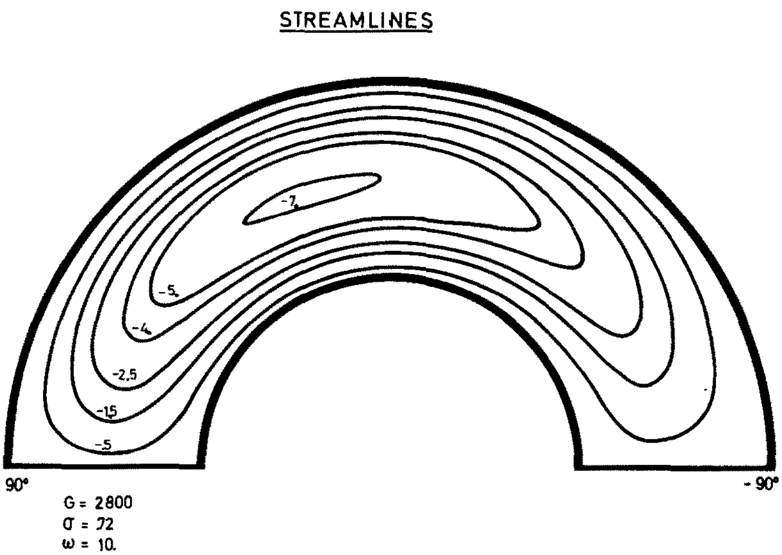

FIG. 4b. Streamlines for $G = 2800$, $\sigma = 0{\cdot}72$ and $\omega = 10$.

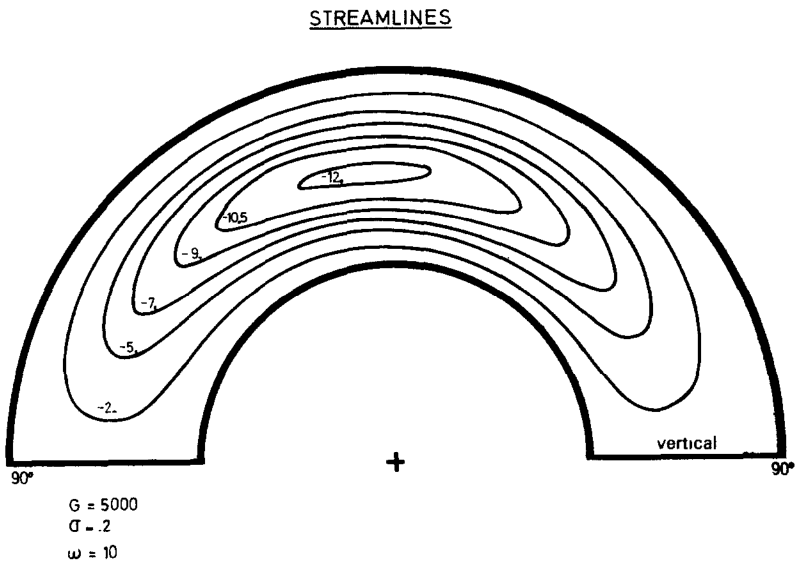

FIG. 5. Streamlines for $G = 5000$. $\sigma = 0{\cdot}20$ and $\omega = 10$.

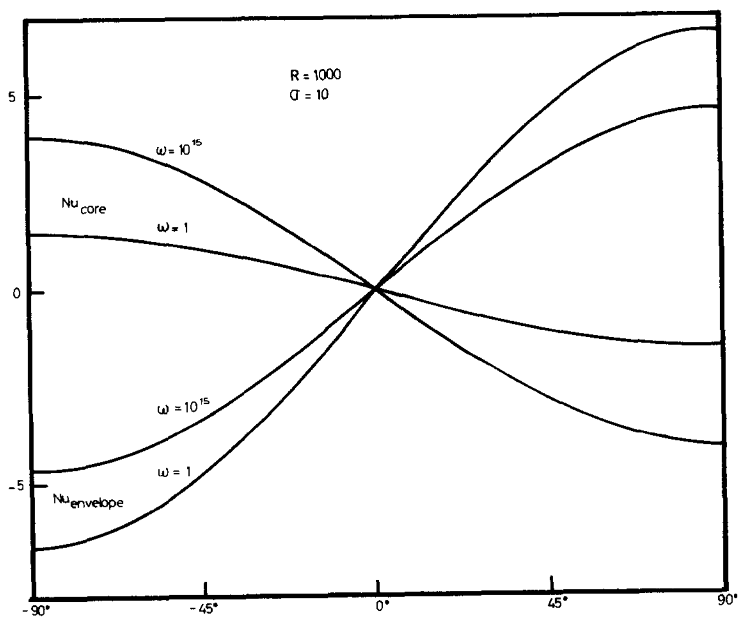

FIG. 6. Nu as a function of θ for $R = 1000$ and $\sigma = 10$ (terms up to order θ_2^a).

Table 1. Local values of the Nusselt number for G = 100. σ = 10·0 *and various values of* ω; inner *signifies core,* outer *signifies envelope,* **—essentially isothermal core. Terms considered: up to and including* θ_2^a, *so as to be quite certain about convergence. This leads to* all *values passing the* 1·000 *line at an angle of* 0° *(horizontal)*

ϑ^o	Nu_{inner}	Nu_{outer}	Nu_{inner}	Nu_{outer}	Nu_{inner}	Nu_{outer}
	$\omega = 1$; $G = 100$; $\sigma = 10{\cdot}0$		$\omega = 10$; $G = 100$; $\sigma = 10{\cdot}0$		$\omega = 10^{15}$; $^*G = 100$; $\sigma = 10{\cdot}0$	
−90	1·15139	0·33328	1·34531	0·48842	1·40277	0·53439
−80	1·14906	0·34345	1·34004	0·49623	1·39662	0·54150
−70	1·14216	0·37363	1·32439	0·51941	1·37838	0·56261
−60	1·13090	0·42290	1·29884	0·55726	1·34860	0·59706
−50	1·11564	0·48975	1·26419	0·60859	1·30820	0·64380
−40	1·09685	0·57211	1·22150	0·67183	1·25843	0·70137
−30	1·07513	0·66745	1·17209	0·74502	1·20082	0·76801
−20	1·05116	0·77287	1·11748	0·82593	1·13713	0·84165
−10	1·02566	0·88513	1·05934	0·91207	1·06931	0·92005
0	0·99944	1·00081	0·99944	1·00081	0·99944	1·00081
10	0·97328	1·11640	0·93961	1·08946	0·92963	1·08148
20	0·94798	1·22838	0·88166	1·17532	0·86201	1·15959
30	0·92431	1·33336	0·82735	1·25579	0·79862	1·23281
40	0·90295	1·42818	0·77830	1·32846	0·74137	1·29891
50	0·88456	1·50997	0·73601	1·39113	0·69199	1·35592
60	0·86966	1·57628	0·70172	1·44193	0·65196	1·40212
70	0·85870	1·62512	0·67647	1·47934	0.62248	1·43615
80	0·85199	1·65512	0·66102	1·50224	0·60443	1·45698
90	0·84973	1·66509	0·65581	1·50995	0·59836	1·46399

Of great interest is the knowledge of the value of the normal temperature gradient at the core boundary. Its negative is the Nusselt number Nu which is a measure of the rate of transport. Some computed values are given in Table 1 and plotted in Fig. 6.

4. DISCUSSION

In the present paper the free convective interaction between a heat dissipating and heat conducting core and a cylindrical fluid envelope is investigated. The expansions used start with a zero-order term which gives a temperature distribution caused by conduction alone, while the first-order term in the expansion for the stream function is obtained through a balance of the forces of buoyancy and of viscous braking in the equations of motion, the non-linear inertia terms being relegated to the subsequent higher-order corrections.† In this respect the expansions used here are of the Stokes type [13]. They should be expected to fail when the radius ratio $\hat{\beta}$ increases without bounds, no matter how small the value of the expansion parameters may be. Indeed, due to the cylindrical geometry of the system, a solution valid in the large (i.e. for all possible values of $\hat{\beta}$) should not even be obtainable for the first expansion term in ψ. This well-known classical result, e.g. [17], was apparently not realized previously in connexion with this type of problem. The physical explanation of this effect is that no matter how small the inertia effects may be, there will always be a sufficiently large distance from the cylinder where *all* terms in the differential equation will be of equal order.

It is now clear that, should $\hat{\beta}$ become large, then the expansions proposed form the "inner" part of a pair of matched asymptotic expansions, to be supplemented by an "outer" or Oseen type of expansion. The reason of the choice of definition of the parameter G which includes the factor $(\hat{\beta} - 1)^3$ is now also clear: it gives the

† This is not the only type of "conjugate" effect; see e.g. [24].

correct scaling for the distance from the cylinder as characteristic dimension when $\hat{\beta}$ becomes large, and causes G to tend to zero when the gap width becomes small, no matter what the temperature difference between the concentric surfaces may be.

On the basis of the expansions proposed no difficulty would seem to arise for vanishly small values of the Prandtl number. However. inspection of the governing equations shows that for this case the energy equation becomes uncoupled from the equation of motion and no higher order terms can be calculated. What is needed is a supplementary transformation which will retain all terms in the energy equation even in the limit as $\sigma \to 0$, will eliminate σ as an indedependent parameter and will nowhere lead to singularities. Such has been obtained e.g. [18]; it is shown that the transfer process takes place in an outer layer where buoyancy and inertia forces balance, with the effect of viscosity relegated to an inner layer which ensures the disappearance of velocity at the fluid/solid interface. The other extreme case, in which σ increases without bounds would render equation (2) *singular*; it would no longer be possible to fulfil *all* the boundary conditions upon the temperature. Again, a further transformation is required which expresses the fact that for the region of heat transfer the viscosity and buoyancy forces balance. A first-order role is played by inertia in an *outer* region which is dragged along by the inner region, where convection is taking place. The outer region forms basically an almost inviscid and isothermal fluid tube. It is interesting to compare this result to Batchelor's prediction [19] of an almost isothermal "fluid core" in confined free convection.

In order not to overburden this paper the perfectly classical stretching transformations involved have not been elaborated here. The main (inner) expansion will be found to proceed in coefficients of R rather than of G: this corresponds to a choice of ψ^*/κ as dimensionless stream function rather than of ψ^*/ν as in the present presentation, equation(4). A peculiar flow polarization above the solid core is also obtained, discussed elsewhere [20].

Lastly, should the conductivity ratio ω vanish then it will become impossible to satisfy both boundary conditions (6) and (7) simultaneously. In effect this case corresponds to an interface of constant heat flux and it is meaningless to evoke "conjugate" effects.

For the qualitative evaluation of the numerical results the most sensitive plots were found to be those of *vorticity* distribution: these will be the first to show evidence of divergence of the series for a given combination of physical parameters. These plots had apparently been little utilized in the evaluation of previous non-conjugate results. The plots obtained here illustrate rather nicely the formation of a comparatively stagnant pool of fluid in the "trough" of the cross-section, where the fluid is most nearly under a quasi-stable density stratification, for certain combinations of the physical parameters. This result is even more striking for the spherical geometry [21].

Reference [16] summarizes some older numerical integration results for the non-conjugate case as well as giving results for the *stability* of the cell flow. Following Torrance [22] and our own experience [23] it would appear that the scheme proposed in [16] should offer only very limited information. Certainly the investigation of flow stability should at best be only indicative. as the principle of exchange of stabilities is apparently unknown to apply to the configuration studied. This, then, severely limits the possibility of *linear* stability analysis.

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CONVECTION LIBRE CONJUGUEE A PARTIR DE CYLINDRES CONDUCTEURS CIRCULAIRES ET HORIZONTAUX

Résumé—Le concept de transfert thermique par convection est habituellement associé à des conditions aux limites spécifiques définies à l'interface entre le solide dissipant ou absorbant la chaleur et le fluide convectif. Dans plusieurs conditions relevant des applications réelles, une telle hypothèse n'est pas justifiée: la distribution de température et de flux sur l'interface lie le champ de température dans le solide au mouvement convectif et à la température du fluide. Cet article considère quelques cas dans lesquels ce couplage est de première importance.

ÜBERLAGERTE FREIE KONVEKTION AN HORIZONTALEN, WÄRMELEITENDEN KREISZYLINDERN

Zusammenfassung—Das Modell der konvektiven Wärmeübertragung ist gewöhnlich verbunden mit eindeutig bestimmten Randbedingungen an der Grenzschicht zwischen dem wärmeabgebenden oder aufnehmenden festen Körper einerseits und dem übertragenden Fluid andererseits.

Unter vielen in der Praxis auftretenden Bedingungen muss eine solche Annahme nicht gerechtfertigt sein: die Temperatur- und Wärmestromverteilung in der Grenzschicht verbindet das Temperaturfeld im festen Körper mit der Bewegung und der Temperatur im Fluid.

Diese Arbeit betrachtet einige Fälle, in denen diese Kopplung von primärer Bedeutung ist.

СОПРЯЖЕННАЯ СВОБОДНАЯ КОНВЕКЦИЯ НА ГОРИЗОНТАЛЬНЫХ ПРОВОДЯЩИХ КРУГЛЫХ ЦИЛИНДРАХ

Аннотация—Понятие конвективного теплообмена обычно связывается с определенными граничными условиями на поверхности раздела между твердым телом и жидкостью. Такое допущение нельзя оправдать для многих условий, возникающих на практике: распределение температуры и теплового потока на поверхности раздела связывает поле температур в твердом теле с конвективным движением и температурой жидкости. В данной работе рассматриваются некоторые случаи, когда эта связь представляет первостепенное значение.